Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Infinitely many monotone Lagrangian tori in del Pezzo surfaces

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Mirror Symmetry and Wall-crossing, UC Berkeley
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Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Definition of monotonicity

Definition

A symplectic manifold (X, ω) is said to be monotone if there exists $K > 0$ such that $\forall H \in \pi_2(X)$

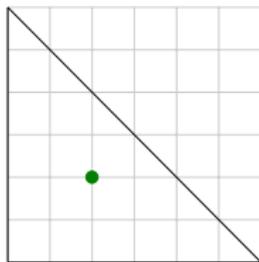$$\int_H \omega = K c_1(H).$$

A Lagrangian $L \subset X$ is said to be monotone if there exists $K_L > 0$ such that $\forall \beta \in \pi_2(X, L)$

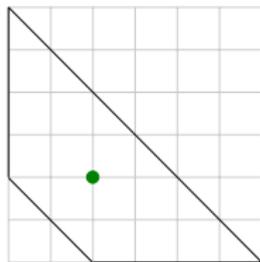$$\int_\beta \omega = K_L \mu(\beta).$$

(If L is orientable then $\mu(\beta) \in 2\mathbb{Z}$.)

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Clifford tori in monotone toric symplectic 4-manifolds (toric del Pezzo surfaces)
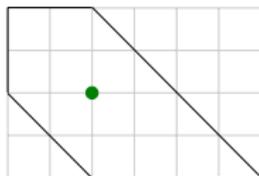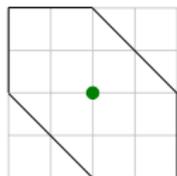

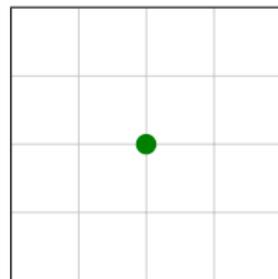$\mathbb{C}P^2$

$\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$

$\mathbb{C}P^1 \times \mathbb{C}P^1$

$\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$

$\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

History

- '94 Chekanov gave the first example of a monotone Lagrangian torus in $\mathbb{C}^n$ not Hamiltonian isotopic (symplectomorphic) to the Clifford tori $\underset{n}{\times} S^1(r)$. Both Clifford and Chekanov tori have manifestations in $\mathbb{CP}^n$, $\underset{n}{\times}\mathbb{CP}^1$.

- '10 Chekanov-Schlenk - Twist tori (new examples in $\mathbb{C}^n$, $\mathbb{CP}^n$, $\underset{n}{\times}\mathbb{CP}^1$, for $n \geq 3$).

- '13 V. - There exists a monotone Lagrangian torus $T(1, 4, 25) \in \mathbb{CP}^2$, bounding 10 families of Maslov index 2 holomorphic disks (total algebraic count 41), therefore not symplectomorphic (Ham. Isot.) to either Clifford or Chekanov tori.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

History

- '14 Auroux - There exists infinitely many monotone Lagrangian tori in $\mathbb{C}^3$.

- '14 V. - There exists infinitely many monotone Lagrangian tori in $\mathbb{CP}^2$.

- '16 V. - There exists infinitely many monotone Lagrangian tori in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k = 3, 4, 5, 6, 7, 8$ and $k = 1$. It is also true for $k = 2$ (by Tonkonog-Pascaleff).

Introduction
Almost toric fibrations - M. Symington
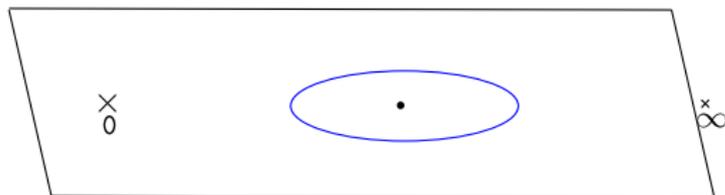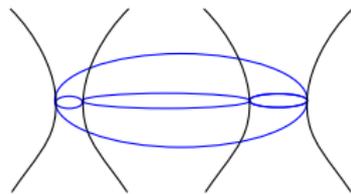Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Description of the count of Maslov index 2 J-holomorphic disks in $\mathbb{CP}^2$

$\mathbb{C}P^2$	$\#\beta$ $\eta_\beta \neq 0$ $\mu(\beta)=2$	$\sum \eta_\beta$ $\mu(\beta)=2$
Clifford $T(1,1,1)$	3	3
Chekanov $T(1,1,4)$	4	5
$T(1,4,25)$	10	41
$T(a^2,b^2,c^2)$	—	—

$a^2 + b^2 + c^2 = 3abc$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Description of the count of Maslov index 2 J-holomorphic disks in $\mathbb{CP}^2$

$\mathbb{C}P^2$	$\#\beta$ $\eta_\beta \neq 0$ $\mu(\beta)=2$	$\sum \eta_\beta$ $\mu(\beta)=2$	convex hull $\{\partial\beta; \eta_\beta \neq 0\}$
Clifford $T(1,1,1)$	3	3	
Chekanov $T(1,1,4)$	4	5	
$T(1,4,25)$	10	41	
$T(a^2,b^2,c^2)$	—	—	

$a^2 + b^2 + c^2 = 3abc$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

D. Auroux's singular Lagrangian fibration:
Clifford type tori $< - >$ Chekanov type tori

$\mathbb{C}P^2$
$\backslash\{(1:0:0),(0:1:0)\}$

$\downarrow \quad [x:y:z]$

$\mathbb{C}P^1 \quad [xy:z^2]$

$e^{i\theta} \cdot [x:y:z] =$
$[e^{i\theta}x : e^{-i\theta}y : z]$

Introduction
Almost toric fibrations - M. Symington
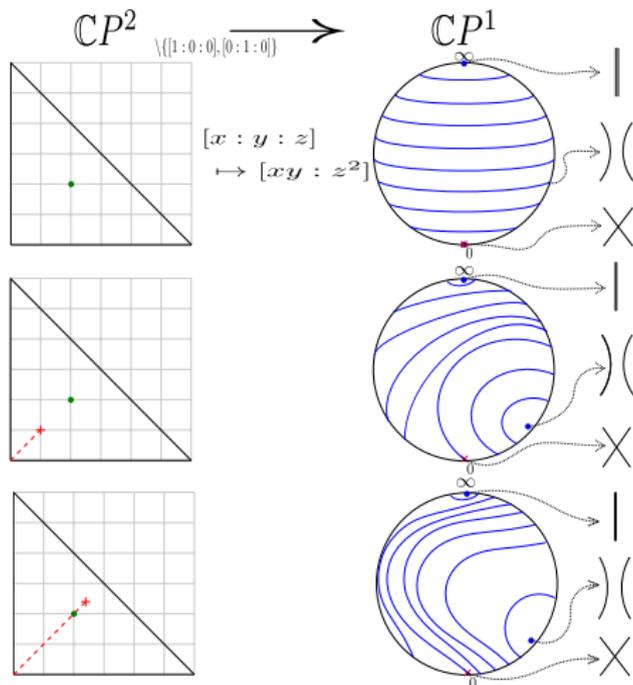Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

D. Auroux's singular Lagrangian fibration:
Clifford type tori $<->$ Chekanov type tori

$\mathbb{C}P^2$
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$\downarrow$ $[x:y:z]$

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$e^{i\theta}\cdot[x:y:z]=$
$[e^{i\theta}x:e^{-i\theta}y:z]$

m_θ

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

D. Auroux's singular Lagrangian fibration:
Clifford type tori $< - >$ Chekanov type tori

$\mathbb{C}P^2$
$\backslash\{(1:0:0),(0:1:0)\}$

$\downarrow \quad [x:y:z]$

$\downarrow$

$\mathbb{C}P^1 \quad [xy:z^2]$

$e^{i\theta} \cdot [x:y:z] =$

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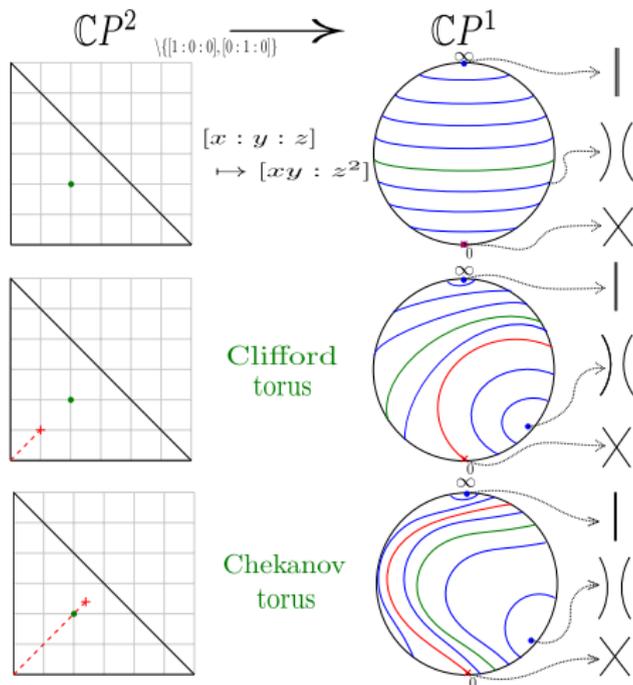Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Almost toric fibrations of $\mathbb{CP}^2$.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Almost toric fibrations of $\mathbb{CP}^2$.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Mutation on almost toric fibrations base diagrams.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
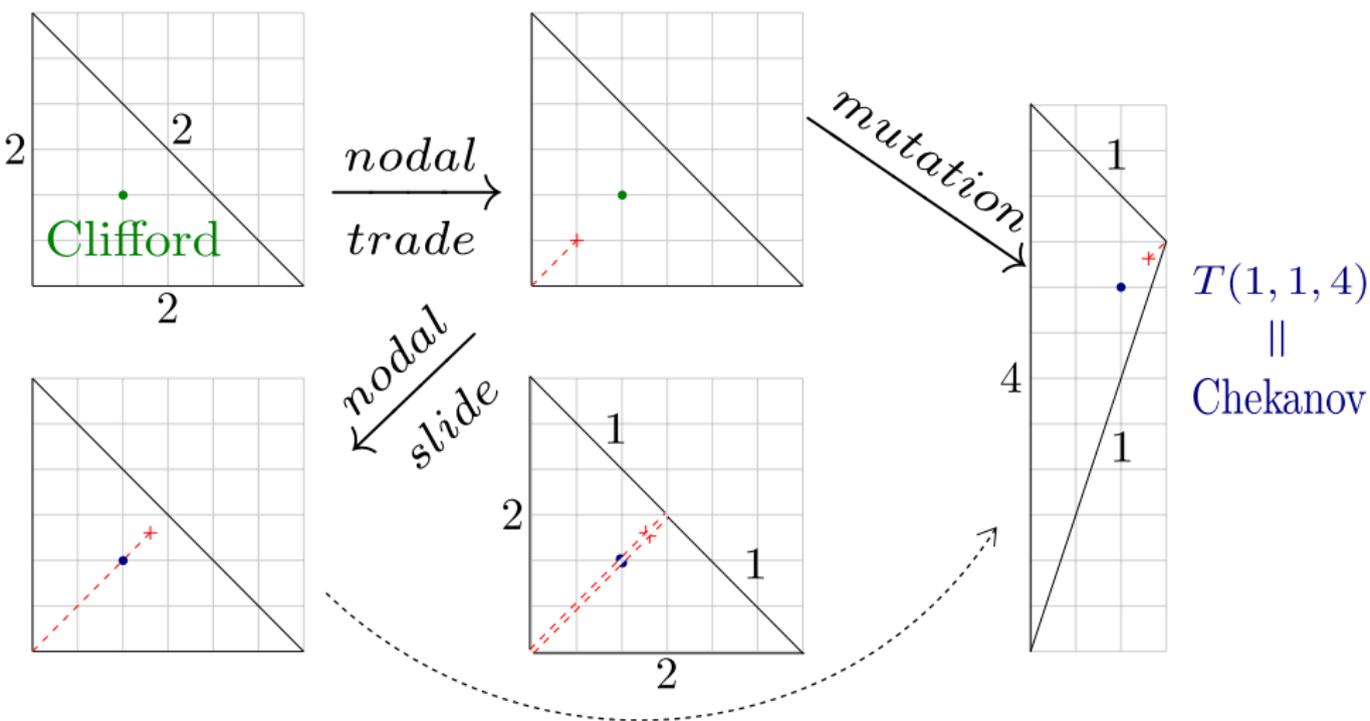
Lemma in $\mathbb{CP}^2$
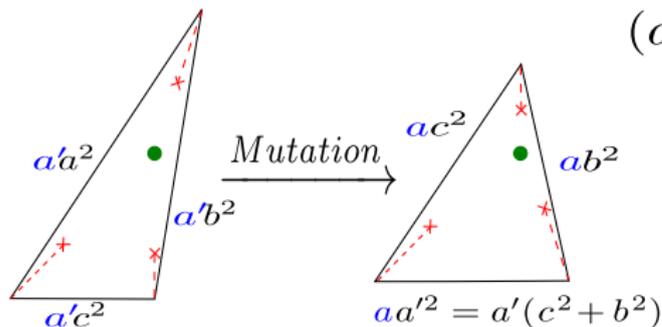
(a, b, c) is a Markov triple

$$a^2 + b^2 + c^2 = 3abc$$

$$b^2 + c^2 = a(3bc - a) = aa'$$

$$a' = (3bc - a)$$

$$a \xrightarrow{mutation} a'$$
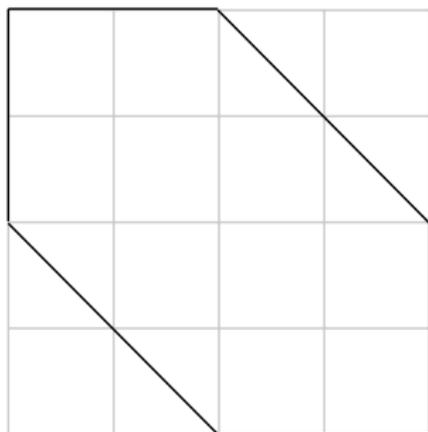
Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Theorem in $\mathbb{CP}^2$

Theorem (V.)

For each Markov triple $\{a, b, c\}$, there exists a monotone Lagrangian torus $T(a^2, b^2, c^2)$ and if $\{a, b, c\} \neq \{d, e, f\}$ then $T(a^2, b^2, c^2)$ is not symplectomorphic (Hamiltonian isotopic) to $T(d^2, e^2, f^2)$.
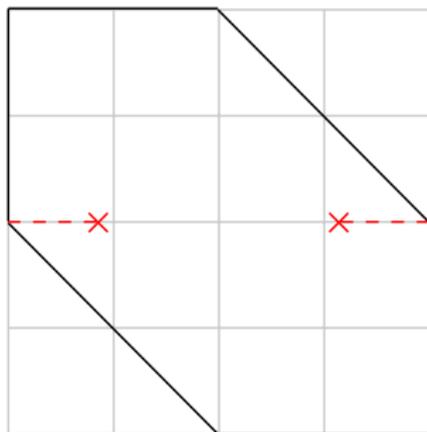
Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

"Creating room" for monotone Blow up

$$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

"Creating room" for monotone Blow up

$$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

"Creating room" for monotone Blow up

$$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

"Creating room" for monotone Blow up

$$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
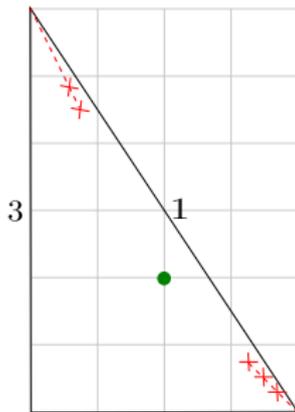Infinitely many tori
Relating to other works

"Creating room" for monotone Blow up

$$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
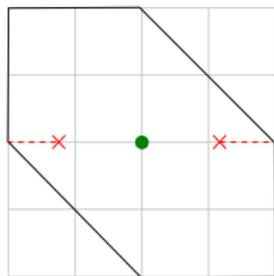Relating to other works

"Creating room" for monotone Blow up

$$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2} \qquad \mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
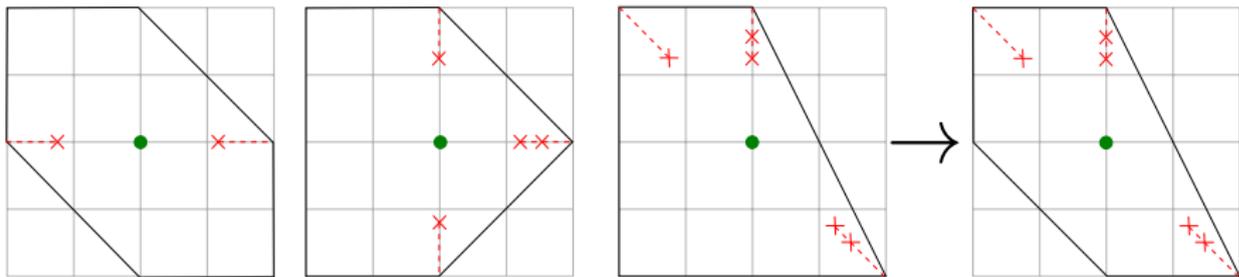
Almost toric fibration on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$

$$a^2 + 2b^2 + 3c^2 = 6abc$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
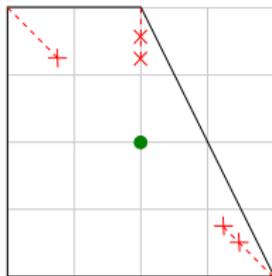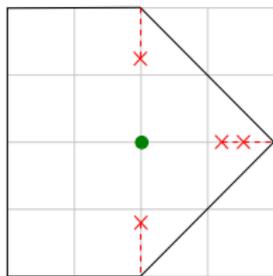
Hacking-Prokhorov Theorem
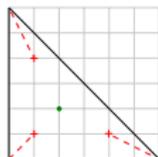
Theorem (Hacking-Prokhorov)

Let X be a weighted projective space, with $K_X^2 \geq 5$. Assume that X admits a ($\mathbb{Q}$-Gorenstein) smoothing to a del Pezzo surface. Then X is of the form:

- $\mathbb{CP}(a^2, b^2, c^2)$, *where* $a^2 + b^2 + c^2 = 3abc$ $(K^2 = 9)$,
- $\mathbb{CP}(a^2, b^2, 2c^2)$, *where* $a^2 + b^2 + 2c^2 = 4abc$ $(K^2 = 8)$,
- $\mathbb{CP}(a^2, 2b^2, 3c^2)$, *where* $a^2 + 2b^2 + 3c^2 = 6abc$ $(K^2 = 6)$,
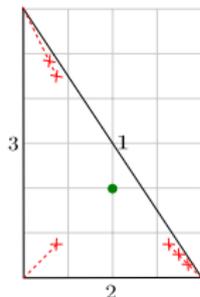- $\mathbb{CP}(a^2, b^2, 5c^2)$, *where* $a^2 + b^2 + 5c^2 = 5abc$ $(K^2 = 5)$.

Moreover, each of the above admits a ($\mathbb{Q}$-Gorenstein) smoothing to a del Pezzo surface.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Almost toric fibrations: triangle type

$\mathbb{C}P^2$

$(I) : p^2 + q^2 + r^2 = 3pqr$
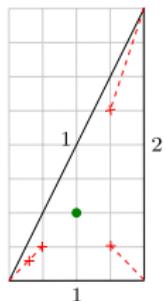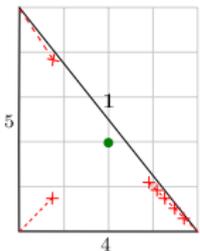$(II) : a^2 + b^2 + c^2 = 3abc$

$\mathbb{C}P^1 \times \mathbb{C}P^1$

$(I) : p^2 + q^2 + 2r^2 = 4pqr$
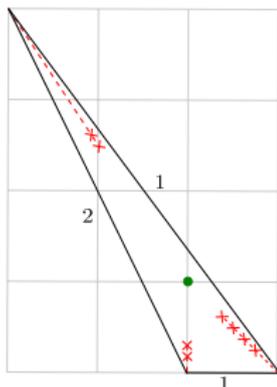$(II) : a^2 + b^2 + 2c^2 = 4abc$

$\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$

$(I) : p^2 + 2q^2 + 3r^2 = 6pqr$
$(II) : a^2 + 2b^2 + 3c^2 = 6abc$

$\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$

$(I) : p^2 + q^2 + 5r^2 = 5pqr$
$(II) : a^2 + b^2 + 5c^2 = 5abc$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Almost toric fibrations: triangle type

$(I) : p^2 + 2q^2 + 6r^2 = 6pqr$
$(II) : a^2 + 2b^2 + 3c^2 = 6abc$

$\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$

$(I) : 2p^2 + 2q^2 + 4r^2 = 8pqr$
$(II) : a^2 + b^2 + 2c^2 = 4abc$

$\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$

$(I) : 3p^2 + 3q^2 + 3r^2 = 9pqr$
$(II) : a^2 + b^2 + c^2 = 3abc$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Almost toric fibrations: triangle type

$(I) : 2p^2 + 2q^2 + 4r^2 = 8pqr$
$(II) : a^2 + b^2 + 2c^2 = 4abc$
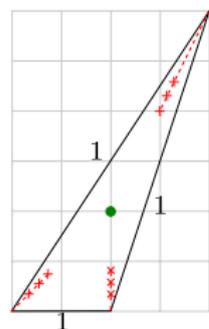
$(I) : p^2 + q^2 + 8r^2 = 4pqr$
$(II) : a^2 + b^2 + 2c^2 = 4abc$

$\mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$

$(I) : p^2 + 3q^2 + 6r^2 = 6pqr$
$(II) : a^2 + 2b^2 + 3c^2 = 6abc$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Almost toric fibrations: triangle type

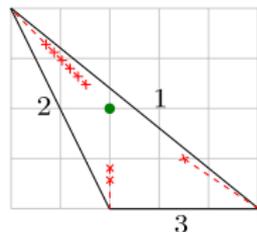
$(I): p^2 + q^2 + 9r^2 = 3pqr$
$(II): a^2 + b^2 + c^2 = 3abc$

$(I): p^2 + 2q^2 + 8r^2 = 4pqr$
$(II): a^2 + b^2 + 2c^2 = 4abc$

$\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$

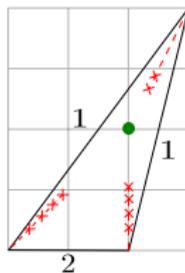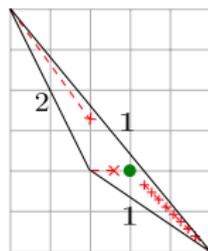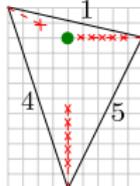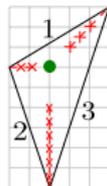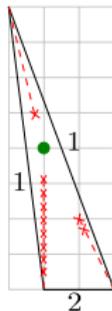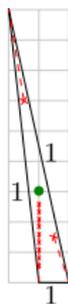$(I): 2p^2 + 3q^2 + 6r^2 = 6pqr$
$(II): a^2 + 2b^2 + 3c^2 = 6abc$

$(I): p^2 + 5q^2 + 5r^2 = 5pqr$
$(II): a^2 + b^2 + 5c^2 = 5abc$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Lemma

Lemma

Consider an almost toric fibration in a "triangular shape". Assume that affine lengths of the edges are proportional to integers of the form k_1a^2, k_2b^2, k_3c^2, so that $k_1a^2 + k_2b^2 + k_3c^2 = Kk_1k_2k_3abc$. Then a mutation with respect to all the nodes opposite to the edge of affine length k_1a^2, has edges with affine lengths proportional to $k_1a'^2$, k_2b^2, k_3c^2, where $a' = KK_2k_3bc - a$.
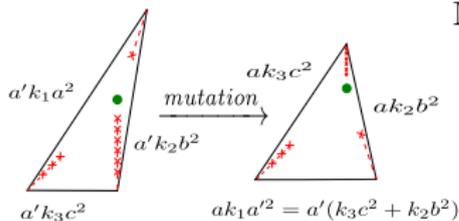
Markov type equation:

$$k_1a^2 + k_2b^2 + k_3c^2 = Kk_1k_2k_3abc$$

$$k_2b^2 + k_3c^2 = k_1aa'$$

$$a' = Kk_2k_3bc - a$$

$$a \xrightarrow{mutation} a'$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Lemma

Markov type equation:

$$k_1 a^2 + k_2 b^2 + k_3 c^2 = K k_1 k_2 k_3 abc$$

$$k_2 b^2 + k_3 c^2 = k_1 a a'$$

$$a' = K k_2 k_3 bc - a$$

$$a \xrightarrow{mutation} a'$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
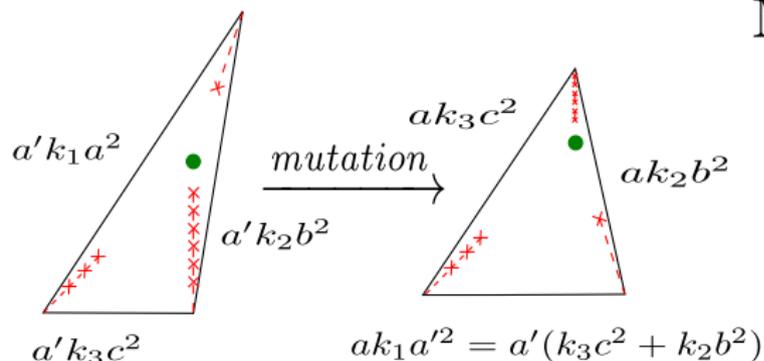
Theorem 1

Theorem

There are infinitely many monotone Lagrangian tori in the del Pezzo surfaces $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, $k = 0, 3, 4, 5, 6, 7, 8$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Sketch of proof.

Consider each of the above almost toric fibration. Use neck-stretching technique to get a degenerate limit of a *J*-holomorphic into the corresponding orbifold. Use positivity of intersection and a result from Cho-Poddar to be able to describe the convex hull the boundary Maslov-2 convex hull (equivalently the convex hull of Newton polytope of the potential function) of the monotone fibre. Show that they are different by comparing affine lengths.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Theorem 1

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
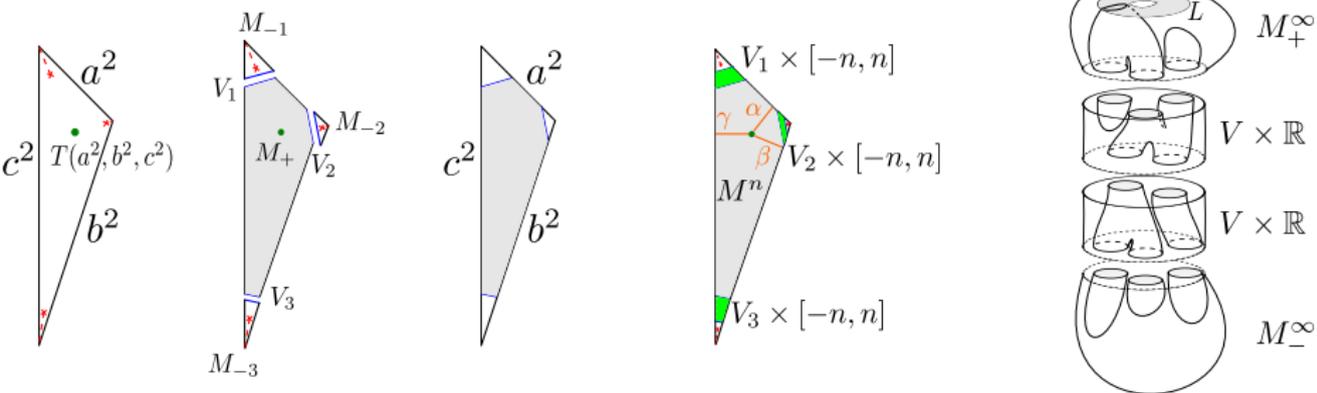
Theorem 2

Theorem

There are infinitely many monotone Lagrangian tori in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
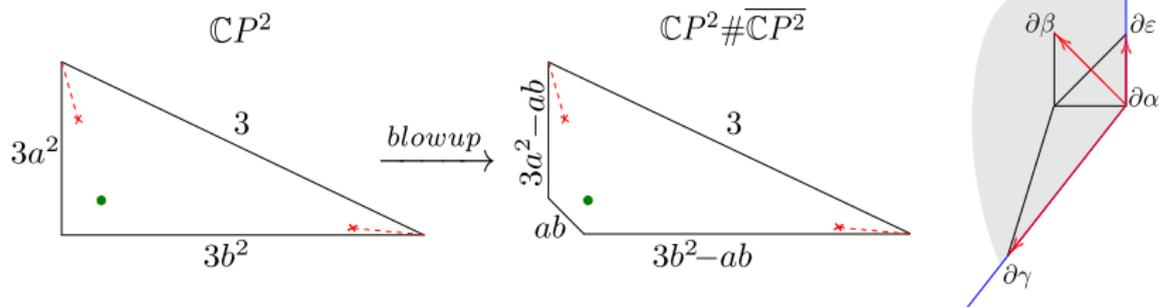
3-block (exceptional) collection

Theorem (Karpov-Nogin)

If there is a 3-block collection $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ on a del Pezzo surface of degree d, so that the exeptional colection $\mathcal{E}_i$ contains n_i sheaves of ranks p, q, r respectively, then they satisfy the Markov type I equation:

$$n_1 p^2 + n_2 q^2 + n_3 r^2 = \sqrt{d n_1 n_2 n_3} pqr \qquad (5.2)$$

Theorem (Karpov-Nogin)

All possible Markov type I equations are the ones apearing on the previous slides.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

m-block (exceptional) collection

Question

Suppose we have an ATBD with node type $((n_1, p_1), \cdots, (n_m, p_m))$ of a del Pezzo surface. Is there an m-block collection $(\mathcal{E}_1, \cdots, \mathcal{E}_m)$, so that the exeptional colection $\mathcal{E}_i$ contains n_i sheaves of rank p_i?

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Tori in Milnor fibres

Proposition (A. Keating)

We have that:

- $T_{3,3,3} = \{x^3 + y^3 + z^3 + 1 = 0\}$ *compactifies to a rank three del Pezzo surface inside* $\mathbb{CP}^3$ *(*$\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$*).*

- $T_{4,4,2} = \{x^4 + y^4 + z^2 + 1 = 0\}$ *compactifies to a rank two del Pezzo surface inside the weighted projective space* $\mathbb{CP}(1,1,2,1)$ *(*$\mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$*).*

- $T_{6,3,2} = \{x^6 + y^3 + z^2 + 1 = 0\}$ *compactifies to a rank one del Pezzo surface inside the weighted projective space* $\mathbb{CP}(1,2,3,1)$ *(*$\mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$*).*

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Tori in Milnor fibres

Remark (A. Keating)

We can get a description of the Milnor fibre $\mathcal{T}_{p,q,r}$ as the result of attaching $p + q + r$ Weinstein handles to D^*T^2. More precisely, we are attaching p, q, and r handles to Legendrian lifts to ∂D^*T^2 of three curves on T^2. We see that each of these three curves in T^2 must intersect the other two once.
In the above proposition, $\mathcal{T}_{p,q,r} = \{x^p + y^q + z^r + 1 = 0\}$
$(p, q, r) = (3, 3, 3); (4, 4, 2); (6, 3, 2)$, respectively.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Lagrangian disks for $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$

On $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2} \cong \{x^3 + y^3 + z^3 + w^3 = 0\} \subset \mathbb{CP}^3$:

$$\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$$

$$a^2 + b^2 + c^2 = 3abc$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works
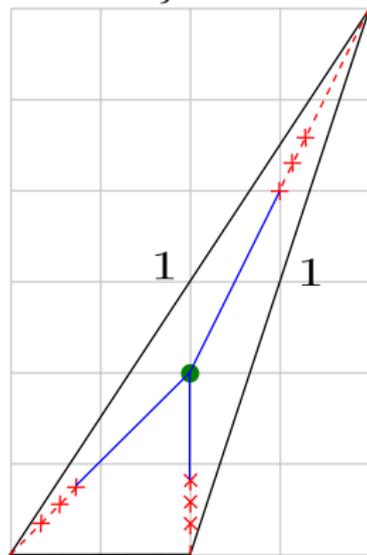
Corollary

Corollary (V.)

There exists infinitely many exact tori on $\mathcal{T}_{p,q,r} =$
$\{x^p + y^q + z^r + 1 = 0\}$ for $(p, q, r) = (3, 3, 3); (4, 4, 2); (6, 3, 2)$,
with respect to the standard symplectic form coming from
attaching an infinite end (half of a symplectization) to its contact
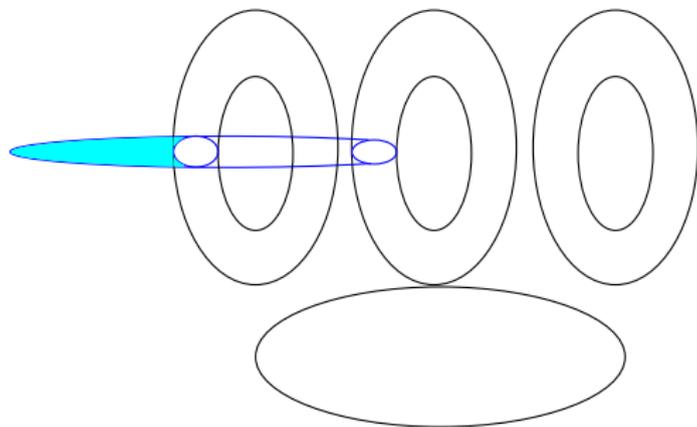*boundary.*

Remark

Note that, there are infinitely many monotone Lagrangian tori
inside a ball, by the result in $\mathbb{CP}^2$. But, by Dmitroglou Rizell's
classification of Lagrangian tori, there are only Clifford and
Chekanov tori in $\mathbb{C}^2$.

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Work of V. Shende, D. Treumann and H. Williams

Attaching handles to D^*T^2, along a Lagrangian disk with boundary in the zero section:

$$D^*T^2$$

Introduction
Almost toric fibrations - M. Symington
Almost toric fibration on del Pezzo surfaces
Infinitely many tori
Relating to other works

Proposition

Proposition (Tonkonog, V.(work in progress))

There exists an almost toric fibration on the Liouville manifold obtained by a Weinstein handle attachment with respect to a Legendrian $S^1 \subset \partial D^* T^2$ corresponding to the conormal lift of an orbit of the T^2 action on $D^* T^2$.

$$D^* T^2$$